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Processing Magnetic Field Data

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ABSTRACT

A method is derived to make a "least squares fit" to measurements of the components of a magnetic field at various points in three dimensions.

To do so, a polynomial model of the field is designed such that the equations

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{B} = 0$$

are satisfied.

This method is obviously equally well applicable to any other field obeying the above equations.

In experiments on high-energy physics one is frequently faced with the problem of deducing the complete magnetic field from measurements of its components at various points.

For the type of magnet used in connection with spark chambers, the field is to be known in a region which is essentially a rectangular box. Therefore we describe the field in Cartesian coordinates.

For our model of the field components we thus choose three polynomials, each in x, y, and z. For the convergence of the series used, it is desirable that the three space coordinates of the volume considered are of the same order. Taking terms with sum of powers less than or equal to N, we have a total of $(N + 3)(N + 2)(N + 1)/2$ terms, being three times the number of combinations of i, j, and k, if $i + j + k \leqslant N$ and i, j, and k are integers $\geqslant 0$.

Due to the constraints

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{B} = 0,$$

the actual number of independent coefficients is $(N + 2)^2 - 1$, which is lower than the number of terms by a factor $(N + 2)/2$. This number can also be derived from Ref. [1].

For $\nabla \times \mathbf{B} = 0$ to be satisfied, it is necessary and sufficient that $\mathbf{B} = \nabla\psi$.

Writing[1]

$$\psi = \sum_{i=1}^{N+2} \sum_{j=1}^{N-i+3} \sum_{k=1}^{N-i-j+4} d(i, j, k)\, x^{i-1} y^{j-1} z^{k-1},$$

we can from this derive expansions for each component, e.g.,

$$B_y = \sum_{i=1}^{N+1} \sum_{j=1}^{N-i+2} \sum_{k=1}^{N-i-j+3} j d(i, j+1, k)\, x^{i-1} y^{j-1} z^{k-1}.$$

From $\nabla \cdot \mathbf{B} = 0$ it follows that $\Delta\psi = 0$ or

$$i(i+1)\, d(i+2, j, k) + j(j+1)\, d(i, j+2, k) + k(k+1)\, d(i, j, k+2) = 0.$$

We now consider this equation as a recurrence relation expressing all d's in terms of those with $i = 1$ and $i = 2$.[2] This can be written explicitly as

$$d(i, j, k) = \sum_{\nu=1}^{\alpha} \frac{(-1)^{\alpha+1} (J-1)!\, (K-1)!\, \binom{\alpha-1}{\nu-1}}{(i-1)!\, (j-1)!\, (k-1)!}\, d(I, J, K)$$

where

$$\alpha = [\![(i+1)/2]\!]; \text{ the largest integer } \leqslant (i+1)/2$$
$$I = i - 2\alpha + 2 \qquad (=1 \text{ or } 2)$$
$$J = j + 2\alpha - 2\nu$$
$$K = k + 2\nu - 2.$$

Throughout this paper we use [[ ]] brackets to denote the truncation of decimals.

The proof of the above expression is readily given by induction; i.e., it is verified for $i = 1$ and $i = 2$, and then it is shown that if it is true for $i = i' - 2$, it follows that it is true for $i = i'$.

We thus find

$$\psi = \sum_{i=1}^{N+2} \sum_{j=1}^{N-i+3} \sum_{k=1}^{N-i-j+4} \sum_{\nu=1}^{\alpha} \frac{(-1)^{\alpha+1} (J-1)!\, (K-1)!\, \binom{\alpha-1}{\nu-1}\, x^{i-1} y^{j-1} z^{k-1}}{(i-1)!\, (j-1)!\, (k-1)!}\, d(I, J, K).$$

[1] For the convenience of FORTRAN writers we use indices from 1 upwards, rather than from 0.

[2] It would be more elegant to treat i, j, and k in a symmetric or rather cyclic fashion. This can indeed be done, but the general expression becomes rather complicated. Doing it either way should, however, give the same final results.

However, to use this further it is necessary to collect the terms with successive independent coefficients $d(I, J, K)$. This results in

$$\psi = \sum_{\substack{I,J,K \\ I=1 \text{ or } 2 \\ 3<I+J+K\leqslant N+4}} \sum_{\nu=1}^{\nu_{\max}} \sum_{\nu'=1}^{\nu'_{\max}} \frac{(-1)^{\alpha+1}\,(J-1)!\,(K-1)!\,\binom{\alpha-1}{\nu-1}\,x^{i-1}y^{j-1}z^{k-1}}{(i-1)!\,(J-1)!\,(k-1)!}\,d(I, J, K)$$

where

$$\alpha = \nu + \nu' - 1, \qquad i = I + 2\alpha - 2,$$

$$j = J - 2\alpha + 2\nu, \qquad k = K - 2\nu + 2,$$

$$\nu_{\max} = \left[\!\left[\frac{K+1}{2}\right]\!\right] \qquad \nu'_{\max} = \left[\!\left[\frac{J+1}{2}\right]\!\right].$$

We have now omitted the term with $i = 1$, $j = 1$, $k = 1$ since $d(1, 1, 1)$ gives no contribution to the expansions of the field components.

This result can also be derived from Ref. [3].

The three indices can be represented by one which is given by the transformation

$$l = I + 2J + (I + J + K - 3)^2 - 3,$$

or inversely

$$I = 2[\![(l - l')/2]\!] - (l - l') + 2,$$

$$J = [\![(l - l' + 1)/2]\!],$$

$$K = [\![l^{1/2}]\!] - I - J + 3,$$

where

$$l' = [\![l^{1/2}]\!]^2 - 1.$$

The $[\![\ ]\!]$ brackets are again used to denote the truncation of decimals.

The maximum value of l is $L = (N + 2)^2 - 1$. However, written in this form we may choose for L, the number of independent coefficients, any integer value.

We now have

$$\psi = \sum_{l=1}^{L} \sum_{\nu=1}^{\nu_{\max}} \sum_{\nu'=1}^{\nu'_{\max}} \frac{(-1)^{\alpha+1}\,(J-1)!\,(K-1)!\,\binom{\alpha-1}{\nu-1}\,x^{i-1}y^{j-1}z^{k-1}}{(i-1)!\,(j-1)!\,(k-1)!}\,d(l).$$

A component of the field at a point (x_m, y_m, z_m) is given by

$$B = \sum_{l=1}^{L} \sum_{\nu=1}^{\nu_{\max}} \sum_{\nu'=1}^{\nu'_{\max}} \frac{(-1)^{\alpha+1}\,(J-1)!\,(K-1)!\,\binom{\alpha-1}{\nu-1}\,x_m^{i-1}y_m^{j-1}z_m^{k-1}}{(i-1)!\,(j-1)!\,(k-1)!}\,d(l),$$

where

I, J, K and α are as before, but

$$i = I + 2\alpha - 2 - \delta_x ,$$

$$j = J - 2\alpha + 2\nu - \delta_y ,$$

$$k = K - 2\nu + 2 - \delta_z ,$$

$$\nu_{\max} = \left[\!\left[\frac{K + 1 - \delta_z}{2}\right]\!\right], \qquad \nu'_{\max} = \left[\!\left[\frac{J + 1 - \delta_y}{2}\right]\!\right],$$

where

$\delta_x = 1$ for x-component and 0 otherwise,

$\delta_y = 1$ for y-component and 0 otherwise,

$\delta_z = 1$ for z-component and 0 otherwise.

This can now be written as

$$B = \sum_{l=1}^{L} A_{ml} d(l)$$

or in matrix notation

$$\mathbf{F} = \mathsf{A}\mathbf{d},$$

where now F is an array of which an element is a particular component of the field at a particular point (x_m , y_m , z_m).

When F contains M observations, the coefficients d are to be chosen so that

$$\sum_{m=1}^{M} \left| F_m - \sum_{l=1}^{L} A_{ml} d(l) \right|^2 W_m \qquad (L \leqslant M)$$

is a minimum; $W_m \propto 1/(\text{statistical error of measurement})^2$

This problem is well known [2] and the solution can be obtained from

$$\mathbf{d} = [\mathsf{A'WA}]^{-1} \mathsf{A'WF},$$

with W = diagonal matrix with the W_m's on the diagonal, and A′ is the transpose of A.

The actual FORTRAN program to solve this was written and tested. A listing can be obtained from the author. The data to be supplied to the program are:

(i) M observations;

(ii) $3 \times M$ coordinates of the points where these observations were taken;

(iii) M numbers between 1 and 3 (incl.) indicating the x, y and z component, respectively;

(iv) the desired value of L.

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